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Computing the Allee Effect and Population Dynamics with Density Dependent Migeration Using Homotopy Perturbation Method

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Abstract: Solutions of nonlinear models are of great importance and their significance has increased a lot. In given paper, the homotopy perturbation method (HPM) is implemented to show the numerical assumption of the population dynamics model with density-dependent migeration and the Allee effect. The resemblance of the numerical solutions attained by HPM with exact solution allows the order of this method. The results show applicability, accuracy and efficiency of HPM in solving the parabolic equation. HPM is effective for solving the transitary non-linear advection diffusion reaction equation.

Keywords: Homotopy Perturbation Method, Allee Effects, Density-Dependent Migeration, Maple 18

1. Introduction

Many physics and engineering problems are modeled by partial differential equations. In many instances these equations are nonlinear and exact solutions are very difficult to obtain. Numerical methods were developed in order to find approximate solutions to these nonlinear equations. However, numerical solutions are insufficient to determine general properties of certain systems of equations and thus analytical methods have been developed.

Homotopy perturbation method was originally introduced by He [1-3]. It is on the base of homotopy technique and traditional perturbation method. In most cases a quick convergent series can get by this method. For numerical purpose with a high degree of accuracy usually a few number of term of the series can be used. This method is effective in partial differential equations in boundary value problems and in different fields. The numerical solution of a equation can obtain without discretization, it is not effected by mulitiple rounding errors and do not require large computer memory and time.

In addition there is no general procedure which is applicable for all such equations. So each equation has to be studied by considering as an individual problem. For this goal, many novel methods for the detection of exact travelling wave solutions of NLEEs have been drawn in huge combinations by a large number of experts. As a result, a lot of work has been done in formulation of several convincing and significant techniques. Different researchers apply these techniques on mathematical and physical models. Such as, the homogenous balance method (Wang, 1995; Zayed, Zedan and Gepreel, 2004), Hirota's bilinear transformation method (Hirota, 1973; Hirota and Satsuma, 1981), auxiliary equation method (Sirendaoreji, 2004), trial function method (Inc and Evans, 2004), Jacobi elliptic function method (Ali, 2011), tanh-function method (Abdou, 2007; Fan, 2000; Malfliet, 1992), homotopy perturbation method (Mohyud-Din et al., 2011), sine-cosine method (Wazwaz, 2004; Bibi and S. T. Mohyud-Din, 2013), truncated Painleve expansion method (Weiss et al., 1983), variational iteration method (He, 1997; Abdou and Soliman, 2005; Abbasbandy, 2007), Exp-function method (He and Wu, 2006; Akbar and Ali, 2012; Naher et al., 2012), (G'/G)-expansion method (Wang et al., 2008;

Akbar et al., 2012; Zayed, 2010; Zayed and Gepreel, 2009; Ali, 2011; Zayed, 2009; Shehata, 2010), improved (G'/G)-expansion method (Zhang, F et al., 2010), exp(-φ(ξ))-expansion method (Khan and Akbar, 2013) and so on.

The purpose of this article to get HPM to reproduce the solutions of the model of population dynamics with density-dependent migeration and the Allee effect. This model can be defined by the transitory non-linear advection-diffusion-reaction equation of the form

$$\frac{\partial V}{\partial T} = -\frac{\partial}{\partial X}\left[\Theta(V)V - D\frac{\partial V}{\partial X}\right] + G(V)V \,,\; X \in \Omega \,,\; T > 0\,. \quad (1)$$

Due to the non linear velocity field $\Theta = \Theta(V)$, V change in space and time. D represent diffusion and G(V) is intrinisc growth rate including all local processes like death, birth and harvesting or predation. Unidentified feild $V = V(\mathrm{Y},\mathrm{T})$ is the population density in $\Omega \subset \Re$ and T.

Spatial distribution is affected by two physical process which are the advection and the isotropic diffusion in model (1). İn order to include the case when the species intentionally move from some specific direction because of chemical communication, for this, we consider a biological mechanism on advection process. These supposition generate the following non- linear velocity field

$$\Theta(V) = \Theta_0 + \Theta_1 V \quad (2)$$

The assumption of model 2 is on the base of density dependent migeration that varies linearly with the population density where Θ_1 depends on the taxis of species. And Θ_0 is density independent migeration velocity which may possibly come from a hydrodynamic solver. We suppose that fluid is not compressible $(div(\Theta_0) = 0)$ and Θ_0, Θ_1 the diffusion coefficient D are constants, generates

$$\frac{\partial V}{\partial T} + (\Theta_0 + 2\Theta_1 V)\frac{\partial V}{\partial X} = D\frac{\partial^2 V}{\partial X^2} + F(V)V \quad (3)$$

Assume that growth dynamics with Allee effect yields

$$G(V)V = \tilde{\beta}U(V - K_0)(K - V) \quad (4)$$

where K is the carrying capacity and K_0 is the measure of the Allee effects. When K is constant, it is convenient to use the dimensionless variable $v = \frac{V}{K}$ so that (4) is re-written as

$$g(v) = \beta v(v - \gamma)(1 - v) \quad (5)$$

when 0 < β < 1 and -1 < β < 0 the weak and strong Alle effect take place. And in eq (5) $\gamma = \mathrm{K}_0/\mathrm{K}$ which denotes the strength of the Allee effects. The parameter normalization constant $\beta = \beta(\gamma)$ defined by maximum strength raet and lead to a family of models. With respect to Allee effect the qualitative results and asymptotic rates of spread are independent from the range of of normalization constant, Using this and $t = \mathrm{T}\beta\mathrm{K}^2$, $x = \mathrm{X}\sqrt{\frac{\beta\mathrm{K}^2}{D}}$, eq (3) can be written in following form

$$v(t) + (\phi_0 + \phi_1 v)v_x = v_{xx} - \gamma v + (1+\gamma)v^2 - v^3 \quad (6).$$

we used the additional dimensionless parameters $\varphi_0 = \frac{\Theta_0}{\mathrm{K}\sqrt{\beta D}}$ and $\varphi_1 = \frac{2\Theta_1}{\sqrt{\beta D}}$. The population densties have been re-scaled so $v \in [0,1]$ in $t \in [0, \mathrm{T}_{final}]$.

Because of assumption of travelling wave solutions so equation (6) is solved in an unbounded domain at infinity conditions which are: For (the species is at its carrying capacity); for (the species is absent) $x \to -\infty \Rightarrow v = o$ for some initial condition. The asymptotic stability analysis of the travelling wave for the scaled diffusion-reaction equation can find by refernces and there provided site in [22].

$$v_t = v_{XX} + h(v) \quad (7)$$

By depending on properties of g wave fronts was derived which are $g(v) = -\gamma v + (1+\gamma)v^2 - v^3$ and has at least two dintict zeros $g_0 = 0$ and $g_1 = 1$ if there exists a strong Allee effect then there will be another zero between g_0 and g_1 at which percapita grwoth rate is zero.

2. Homotopy Perturbation Method HPM

For a general nonlinear boundary-conditioned differential equation

$$A(u) - f(r) = 0, r \in \Omega \quad (8)$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, u is the r-dependent unknown function is the boundary of the domain Ω.

The operator A can, generally speaking, be divided in to two parts L and N, where L is linear, and N is nonlinear, therefore

Eq. (8) can be written as

$$L(u) + N(u) - f(r) = 0 \quad (9)$$

By using the homotopy technique, one can construct a homotopy $V(r,p): \Omega \times [0,1] \to A$ which satisfies

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0 \quad (10)$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[A(v) - f(r)\right] = 0 \quad (11)$$

Where $p \in [0,1]$ is an embedding parameter, and u_0 is the initial approximation which satisfies the boundary conditions

Clearly,

$$H(v,0) = L(v) - L(u_0) = 0 \tag{12}$$

and

$$H(v,1) = A(v) - f(r) = 0 \tag{13}$$

The changing process of p from zero to unity is just that of $V(r,p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If the embedding parameter p; $(0 \le p \le 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of Eq. (10) can be given as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{14}$$

and after setting p = 1 results in the approximate solution of Eq. (8) as

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \ldots$$

3. Implementation of HPM

Here HPM will apply to the given non linear patial differential equation of the form:

$$v_t + (\phi_0 + \phi_1 v) v_x = v_{xx} - \gamma v + (1+\gamma) v^2 - v^3 \tag{15}$$

And initial conditions are

$$v(x,0) = \frac{\gamma \exp(\mu_1 \nu_1) + \exp(\mu_2 \nu_2)}{1 + \exp(\mu_1 \nu_1) + \exp(\mu_2 \nu_2)}$$

Where $\nu_i = x + c_i$, $i = 1,2$, $\mu_1 = \gamma/2$ and $\mu_2 = 1/\sqrt{2}$ and c_1, c_2 are arbitrary constants.

we have the following homotopy

$$v_t + p\left\{(\phi_0 + \phi_1 v) v_x - v_{xx} + \gamma v - (1+\gamma) v^2 + v^3\right\} = 0 \tag{16}$$

consider the eq (16) has solution in this form:

$$v(x,t) = v_0(x,t) + pv_1(x,t) + p^2 v_2(x,t) + p^3 v_3(x,t) + \ldots \tag{17}$$

Substituting eq (17) into eq (16) and write the sum of same poers of p which gives following:

$$p^0 : (v_0)_t = 0$$

$$p^1 : (v_1)_t + \phi_0 (v_0)_x + \phi_1 (v_0)(v_0)_x - (v_0)_{xx} + \gamma v_0 - (1+\gamma) v_0^2 + v_0^3 = 0$$

$$p^2 : (v_2)_t + \phi_0 (v_1)_x + \phi_1 (v_1)(v_0)_x + \phi_1 (v_0)(v_1)_x - (v_1)_{xx} + \gamma v_1 - 2(1+\gamma) v_0 v_1 + 3 v_0^2 v_1 = 0$$

$$p^3 : (v_3)_t + \phi_0 (v_2)_x + \frac{1}{2}\phi_1 \left(2v_2 (v_0)_x + 2v_1 (v_1)_x + 2v_0 (v_2)_x\right) - (v_2)_{xx} + \gamma v_2 - \frac{1}{2}(1+\gamma)\left(2(v_1)^2 + 4v_0 v_2\right) - \frac{1}{2}\left(-6v_0 (v_1)^2 - 6(v_0)^2 v_2\right) = 0$$

$$\cdots$$

We obtain

$$v_1(x,t) = \frac{1}{\left(1 + \exp(\mu_1 \nu_1) + \exp(\mu_2 \nu_2)\right)^3}\left[t\left[\begin{array}{l} \gamma \exp(\mu_1 \nu_1)\left(-\gamma + \mu_1^2\right) - \exp(2\mu_1 \nu_1)\gamma\left(\gamma - \gamma^2 + \mu_1^2\right) \\ -\exp(2\mu_1 \nu_1 + \mu_2 \nu_2)(-1+\gamma)\left(\gamma - \gamma^2 + (\mu_1 - \mu_2)^2\right) \\ +\exp(\mu_2 \nu_2)\left(\gamma - \mu_2^2\right) - \exp(2\mu_2 \nu_2)\left(-1 + \gamma + \mu_2^2\right) \\ +\exp(\mu_1 \nu_1 + 2\mu_2 \nu_2)(-1+\gamma)\left(-1 + \gamma + \mu_1^2 - 2\mu_1 \mu_2 + \mu_2^2\right) \\ +\exp(\mu_1 \nu_1 + \mu_2 \nu_2)\left((-1+2\gamma)\mu_1^2 - 2(1+\gamma)\mu_1 \mu_2 - (-2+\gamma)\mu_2^2\right) \end{array}\right]\right]$$

and so on by ($\theta_0 = \theta_1 = 0$) the other terms can obtain

3.1. Special Cases of the Model

Case 1 No Migration

When no Migration

$$\varphi_0 = \varphi_1 = 0$$

and (6) reduces to

$$v_t = v_{XX} - \gamma v + (1+\gamma) v^2 - v^3 \tag{18}$$

The exact solution of (18) is

$$v(x,t)=\frac{\gamma\exp(\mu_1\nu_1)+\exp(\mu_2\nu_2)}{1+\exp(\mu_1\nu_1)+\exp(\mu_2\nu_2)}.$$

where g_1 $\nu_i = x-\eta_i t + c_i$, $i=1, 2,$, $\eta_i=\sqrt{2}(1+\gamma)-3\mu_i$, $\mu_1=\gamma/\sqrt{2}$ and $\mu_2=1/\sqrt{2}$ and c_1 and c_2 are arbitrary constants.

Case II. Density-independent migeration

This case indicate that speed of the migeration of species does not depend on the density of population, for example the dynamics of the population when drifting with wind are described by the following equation

$$v_t+\phi_0 v_x = v_{XX}-\gamma v+(1+\gamma)v^2-v^3 \tag{19}$$

where φ_0 represent the speed of advection. By assumption traveling wave coordinates, $(x,t)\to(z,t)$ where $z=x-\varphi_0 t$, $v=\tilde{v}(z,t)$ from equation (19)

$$\tilde{v}_t=\tilde{v}_{XX}-\gamma\tilde{v}+(1+\gamma)\tilde{v}^2-\tilde{v}^3 \tag{20}$$

Equation (18) coincides with equation (20) and equation (18) gives an exact solution of equation (20) with the change of $x\to z$

Case III. Density-Dependent Migration

We assume the case that migeration occur because of biological mechanisms which are considered as density dependent and in our assumption environmental factors are absent which are reason of density independent advection. The $\varphi_0=0$ and from equation (6) the we have following equation

$$v_t+\phi_1 v v_x = v_{XX}-\gamma v+(1+\gamma)v^2-v^3 \tag{21}$$

the exact solution of equation (21) is

$$v(x,t)=\frac{\gamma\exp(\omega_1\nu_1)+\exp(\omega_2\nu_2)}{1+\exp(\omega_1\nu_1)+\exp(\omega_2\nu_2)}$$

Case IV. General Case

Mostly migeration occur because of density dependent and density independent factors. The dynamics of a knwon population are then defined by full (6) where no $\varphi_0\neq 0$ and $\varphi_1\neq 0$. İn this case the exact solution is

$$v(x,t)=\frac{\gamma\exp\big(\omega_1\big(x-(q_1+\phi_0)t+\varepsilon_1\big)\big)+\exp\big(\omega_2\big(x-(q_2+\phi_0)t+\varepsilon_2\big)\big)}{1+\exp\big(\omega_1\big(x-(q_1+\phi_0)t+\varepsilon_1\big)\big)+\exp\big(\omega_2\big(x-(q_2+\phi_0)t+\varepsilon_2\big)\big)}$$

This exact solution and eqution (21) has same notations

The error between the exact soltion and both method of approximate solution of the given four cases are shown in figure 1-4.

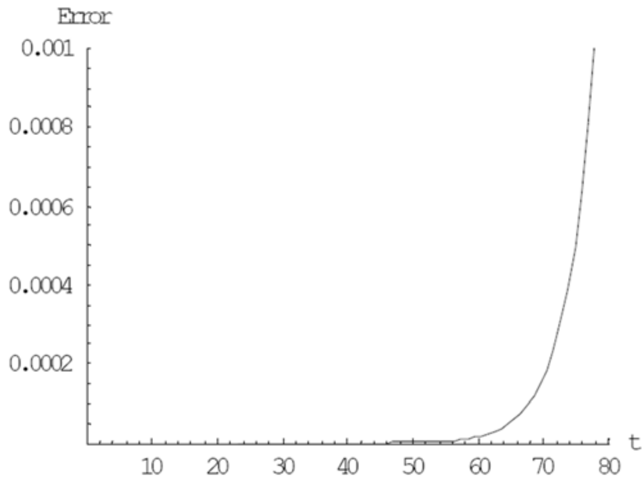

Fig. 1. *(Case 1) The error at* $x_0=20$ *and* $\varphi_0=0, \varphi_1=0$.

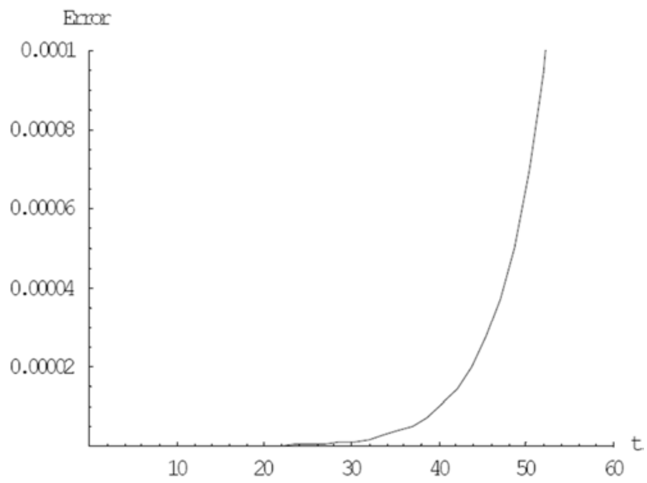

Fig. 2. *(Case 2) The error at* $x_0=65$ *and* $\varphi_0=0.1, \varphi_1=0$.

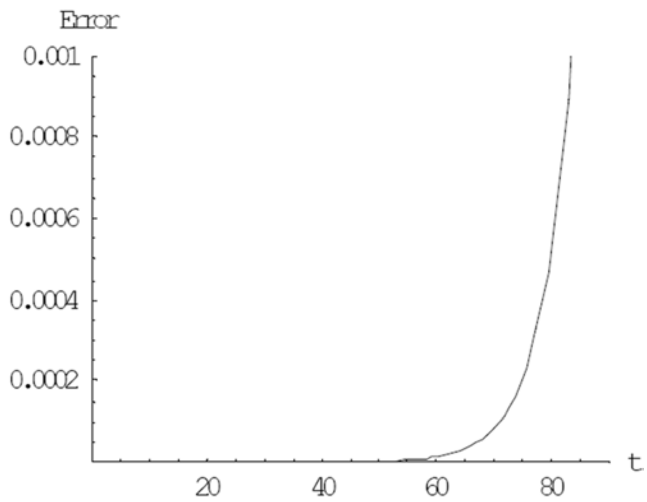

Fig. 3. *(Case 3) The error at* $x_0=50$ *and* $\varphi_0=0, \varphi_1=0.1$.

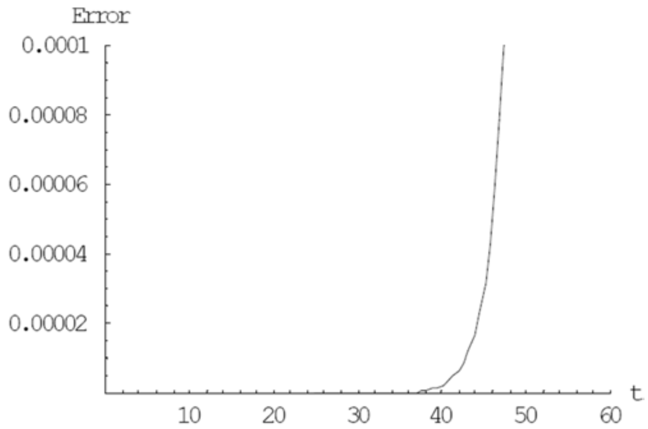

Fig. 4. *(Case 4) The error at* $x_0 = 50$ *and* $\varphi_0 = -\varphi_1 = -0.1$.

4. Conclusion

Homoptopy perturbation method is used to find th approximate and exact soltion of nonlinear equation effectively without linearlization or transforming the equation with high accuracy. İn this paper Homotopy perturbation method is used to solve the Allee effect and model of population dynamics with density dependent migeration. İn which Numeriacl approximation indicate a high degree of accuracy, and in few terms this approximation is accurate which clear the advantage of this method. By taking new terms of iteration formulas these errors can be reduce.

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